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# Polar decomposition of oblique projections

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The partial isometries and the positive semidefinite operators which appear as factors of polar decompositions of bounded linear idempotent operators in a Hilbert space are characterized.



## 1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. The polar decomposition of $T \in L(\mathcal{H})$ is the unique factorization $T = V_T A_T$, where $V_T$ is a partial isometry, $A_T$ is a positive semidefinite operator and $N(V_T) = N(A_T)$ (here, $N$ denotes the nullspace).

This paper is devoted to the study of the polar factors of an oblique projection $Q$, i.e., an idempotent $Q \in L(\mathcal{H})$. More precisely, denote by $\mathcal{J}$ the set of all partial isometries on $\mathcal{H}$, $L(\mathcal{H})^+$ the cone of all positive semidefinite operators on $\mathcal{H}$, and $\mathcal{Q}$ the set of all idempotents of $L(\mathcal{H})$. Our main goal is to characterize the sets

$$\mathcal{J}_{\mathcal{Q}} = \{V \in \mathcal{J} : \text{there exists } Q \in \mathcal{Q} \text{ such that } V = V_Q\}$$

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and

$$L(\mathcal{H})^+_{\mathcal{Q}} = \{A \in L(\mathcal{H})^+ : \text{there exists } Q \in \mathcal{Q} \text{ such that } A = A_Q\}.$$

It is well-known that for every $T \in L(\mathcal{H})$ it holds $A_T = |T| = (T^*T)^{1/2}$. However, there is no formula for $V_T$, in general. We prove that for $Q \in \mathcal{Q}$ both $|Q|$ and $V_Q$ have an explicit expression, and they form a relatively regular pair, in the sense that $|Q|V_Q|Q| = |Q|$ and $V_Q|Q|V_Q = V_Q$; moreover, this property characterizes the idempotency of $Q = V_Q|Q|$.

For any closed subspace $\mathcal{S}$ denote by $P_{\mathcal{S}}$ the orthogonal projection onto $\mathcal{S}$. It is known that the mapping $T \longrightarrow P_{R(T)}$ is not continuous with respect to the norm (uniform) topology. However, the restriction to $\mathcal{Q}$ is Lipschitz with constant 1, by a result of Kato [14, Theorem 6.35, p. 58]. From this, it also follows that the mapping $Q \longrightarrow V_Q$ is continuous, in contrast with the fact that the mapping $T \longrightarrow V_T$ is not. This result is related to the fact that the mapping $Q \longrightarrow Q^\dagger$ is Lipschitz of constant 2 while, in general, $T \longrightarrow T^\dagger$ is not continuous; here $^\dagger$ denotes the Moore–Penrose pseudoinverse [8].

The main results of the paper are the characterizations

$$\mathcal{J}_{\mathcal{Q}} = \{V \in \mathcal{J} : VP_{R(V)} \in L(\mathcal{H})^+, R(VP_{R(V)}) = R(V)\}$$

and

$$L(\mathcal{H})^+_{\mathcal{Q}} = \{A \in L(\mathcal{H})^+ : \gamma(A) \geqslant 1,\ \dim \overline{R(A - P_{R(A)})} \leqslant \dim N(A)\}.$$

We also prove that the map $Q \longrightarrow V_Q$ is injective with inverse $V \longrightarrow (V^2V^*)^\dagger V$ and we characterize, for each $A \in L(\mathcal{H})^+$, the set

$$\{Q \in \mathcal{Q} : |Q| = A\}.$$

We also show that the map $Q \longrightarrow (QQ^*, Q^*Q)$ is injective and we characterize its image. More precisely, it consists of all pairs $(A, B) \in L(\mathcal{H})^+ \times L(\mathcal{H})^+$ such that $P_{R(A)}BP_{R(A)} = P_{R(A)}$ and $P_{R(B)}AP_{R(B)} = P_{R(B)}$.

## 2. Preliminaries

### 2.1. Polar decompositions

Given $T \in L(\mathcal{H})$, there exists a unique partial isometry $V$ and a unique positive (semidefinite) operator $A$ such that $T = VA$ and $N(V) = N(A) = N(T)$. The operator $A$ is exactly $|T| = (T^*T)^{1/2}$. However, in general there is no explicit formula for $V$. The following equalities hold: $T = |T^*|V;\ |T| = V^*T;\ T|T|^\dagger = V$ if $T$ has a closed range. In this last case, the Moore–Penrose inverse $T^\dagger$ can be obtained by functional calculus and $T^\dagger$ belongs to the $C^*$-algebra generated by $T$. It should be noticed that in matrix analysis literature, in the definition of polar decompositions many times there is no condition on $N(V)$, so that there are many "polar decompositions" of an operator $T$ (see the comments by Higham [11, p. 194]). Observe that the canonical polar decomposition $T = V|T|$, with $N(V) = N(T)$, can be changed to $T = U|T|$, with a unitary $U$, if the index of $T$ is zero, i.e., if $\dim N(T) = \dim N(T^*)$. This is the case of every projection $Q$.

### 2.2. Reduced minimum modulus

The *reduced minimum modulus* of $T \in L(\mathcal{H})$ is the number $\gamma(T) = \inf\{\|T\xi\| : \xi \in N(T)^\perp, \|\xi\| = 1\}$. It is well known that $\gamma(T) = \gamma(T^*) = \gamma(|T|) = \gamma(T^*T)^{1/2}$, and $\gamma(T) > 0$ if and only if $T$ has closed range. Indeed, it holds $\|T^\dagger\| = 1/\gamma(T)$ if $T$ has closed range (see [5]; [14, p. 231]).

### 2.3. Comparison of oblique projections

The next result is widely used in the next sections. Its proof is elementary and will be omitted.

**Lemma 2.1.** *Let $P$, $Q$ be two oblique projections. Then:*

1. $PQ = Q \iff R(Q) \subseteq R(P)$;
2. $PQ = P \iff N(Q) \subseteq N(P)$;
3. $P = Q \iff N(P) = N(Q)$ *and* $R(P) = R(Q) \iff N(Q) \subseteq N(P)$ *and* $R(Q) \subseteq R(P)$.

We frequently use, without mention, the fact that there is a natural bijective correspondence between the set $\mathcal{Q}$ of all oblique projections in $\mathcal{H}$ and the set of direct sum decompositions $\mathcal{W} \dotplus \mathcal{M} = \mathcal{H}$. This bijection associates to each decomposition $\mathcal{W} \dotplus \mathcal{M} = \mathcal{H}$ the oblique projection $Q = P_{\mathcal{W}//\mathcal{M}}$ with range $\mathcal{W}$ and null space $\mathcal{M}$.

## 3. The polar factors of an oblique projection

We start with a series of lemmas which shows that each one of the partial isometry and the absolute value of an oblique projection is a generalized inverse of the other.

**Lemma 3.1.** *Let $Q$ be an oblique projection. Then*

$$V_Q|Q|V_Q = V_Q.$$

**Proof.** From $Q^2 = Q$ and $Q = V_Q|Q|$ we get $V_Q|Q|V_Q|Q| = V_Q|Q|$, i.e., $V_Q|Q|V_Q = V_Q$ on $R(|Q|) = R(Q^*) = N(Q)^\perp$. But $V_Q|Q|V_Q$ and $V_Q$ obviously coincide on $N(Q)$, because $N(V_Q) = N(Q)$. Thus, $V_Q|Q|V_Q = V_Q$ on $\mathcal{H}$. $\square$

**Lemma 3.2.** *Let $Q$ be an oblique projection. Then*

$$|Q|V_Q = V_Q^*|Q| = P_{N(Q)^\perp}.$$

**Proof.** By Lemma 3.1, it follows that $|Q|V_Q$ is an idempotent. The chain of inclusions $N(Q) = N(V_Q) \subseteq N(|Q|V_Q) \subseteq N(V_Q|Q|V_Q) = N(V_Q) = N(Q)$ implies that $N(|Q|V_Q) = N(Q)$. On the other hand, $R(|Q|V_Q) \subseteq R(|Q|) = N(Q)^\perp$. Therefore, $|Q|V_Q$ is an oblique projection with the same nullspace as $P_{N(Q)^\perp}$ and whose range is contained in $N(Q)^\perp$. Then $|Q|V_Q = P_{N(Q)^\perp}$, by Lemma 2.1. By taking adjoints we get $V_T^*|Q| = P_{N(Q)^\perp}$. $\square$

**Remark 3.3.** If $T \in L(\mathcal{H})$ has polar decomposition $V_T|T|$, then the operator $T_0 = |T|V_T$ is called the Duggal (or Duggal-Porta) transform of $T$. Lemma 3.2 says that the Duggal transform of $Q \in \mathcal{Q}$ is $P_{N(Q)^\perp}$. We will extend this result to the family of Aluthge transforms at the end of this section.

**Lemma 3.4.** *Let $Q$ be an oblique projection. Then*

$$V_Q = P_{R(Q)}|Q|.$$

**Proof.** It suffices to combine the last two results: $V_Q = V_Q|Q|V_Q = V_Q(V_Q^*|Q|) = P_{R(Q)}|Q|$. $\square$

**Lemma 3.5.** *Let $Q$ be an oblique projection. Then*

$$Q = P_{R(Q)}Q^*Q.$$

**Proof.** By Lemma 3.4, it holds $Q = V_Q|Q| = P_{R(Q)}|Q|^2 = P_{R(Q)}Q^*Q$. $\square$

**Lemma 3.6.** *Let $Q$ be an oblique projection. Then*

$$|Q|V_Q|Q| = |Q|.$$

**Proof.** By Lemmas 3.4 and 3.5, it holds $V_Q|Q| = P_{R(Q)}|Q|^2 = Q$; thus, $|Q|V_Q|Q| = |Q|Q$. Observe now that $|Q|Q = |Q|$ on $R(Q)$ and on $N(Q)$, so we get the result. □

For later reference we state the following lemma.

**Lemma 3.7.** *For any oblique projection $Q$, the positive part and the partial isometry part of $Q^*$ are related to those of $Q$ in such a way that $|Q^*| = V_Q|Q|V_Q^*$, $V_{Q^*} = V_Q^*$ and $Q = |Q^*|V_Q$.*

We collect these results, and their analogous for the reverse polar decomposition, in the next statement.

**Theorem 3.8.** *Given an oblique projection $Q \in L(\mathcal{H})$ with polar decompositions $Q = V_Q|Q| = |Q^*|V_Q$, the following identities hold*:

1. $V_Q = P_{R(Q)}|Q| = |Q^*|P_{N(Q)^\perp}$;
2. $V_Q|Q|V_Q = V_Q = V_Q|Q^*|V_Q$;
3. $|Q|V_Q|Q| = |Q|$ *and* $|Q^*|V_Q|Q^*| = |Q^*|$;
4. $|Q|V_Q = V_Q^*|Q| = P_{N(Q)^\perp}$ *and* $V_Q|Q^*| = |Q^*|V_Q^* = P_{R(Q)}$;
5. $P_{R(Q)}Q^*Q = Q = QQ^*P_{N(Q)^\perp}$.

**Proof.** The first identity of each 1, 2, 3 and 4 follows directly from Lemmas 3.4, 3.1 and 3.6. The second identities can be easily derived by using Lemma 3.7. □

**Corollary 3.9.** *The mapping $Q \longrightarrow V_Q$ is continuous with respect to the operator (uniform) topology.*

**Proof.** By a result of Kato [14, Theorem 6.35, p. 58], $\|P_{R(Q)} - P_{R(Q')}\| \leqslant \|Q - Q'\|$ for every $Q, Q' \in \mathcal{Q}$. The continuity of $T \longrightarrow |T|$ is well known and holds not only on $\mathcal{Q}$ but on $L(\mathcal{H})$. Therefore, the factorization $V_Q = P_{R(Q)}|Q|$ proves the result. □

**Remark 3.10.** (1) Since $P_{R(Q)}$ and $Q$ are idempotents with the same range, by Lemma 2.2 it follows that $P_{R(Q)}Q = Q$ and $QP_{R(Q)} = P_{R(Q)}$, so that $P_{R(Q)}Q^*Q = P_{R(Q)}Q = Q$.

(2) The decomposition of Lemma 3.4 is a polar decomposition, in the sense that $|Q|$ is a positive semidefinite operator and $P_{R(Q)}$ is a partial isometry. However, the nullspace condition does not hold and, of course, the positive factor is not $|X|$ in either case $X = V_Q, V_Q^*$. Higham [11] suggests the name of "canonical polar factorization" for the one we are using. Observe that, in general, the literature in matrix computations is not uniform in this respect.

(3) Given $Q \in \mathcal{Q}$, it is well known [9] that the orthogonal projection $P_{R(Q)}$ can be explicitly obtained from $Q$ by means of the formula $P_{R(Q)} = QQ^*(I - (Q - Q^*)^2)^{-1}$. We present a short proof of this fact: observe first that $I - (Q - Q^*)^2 = I + (Q - Q^*)^*(Q - Q^*)$ is positive and invertible. Also using Lemma 2.1 several times we get $P_{R(Q)}(I - (Q - Q^*)^2) = P_{R(Q)}(I - Q - Q^* + QQ^* + Q^*Q) = QQ^*$.

Observe also that $QQ^* = P_{R(Q)}(I - (Q - Q^*)^2)$ has some of the features of a polar decomposition in the sense that $P_{R(Q)}$ is a partial isometry with the same nullspace as $QQ^*$ and $I - (Q - Q^*)^2$ is positive. However, this is not the polar decomposition of $QQ^*$. In fact, the operator $I - (Q - Q^*)^2$ has a trivial nullspace. In order to get the polar decomposition of $QQ^*$, it suffices to observe the identity $QQ^* = P_{R(Q)}QQ^*$ and verify that $P_{R(Q)}$ and $QQ^*$ satisfy the nullspace condition. In general, if $A$ is a positive (semidefinite) operator then its polar decomposition is provided by the identity $A = P_{R(A)}A$.

It is well-known that the study of projections is closely related to the study of diverse types of generalized inverses. The sets $S = \{(A, B) : A, B \in L(\mathcal{H}), ABA = A, BAB = B\}$ and $S_Q = \{(A, B) : A, B \in L(\mathcal{H}), AQ = A, QB = B, BA = Q\}$, for a fixed $Q \in \mathcal{Q}$, have been studied from a geometrical point of view in [3,7], respectively. Notice that $S = \cup_{Q\in\mathcal{Q}} S_Q$. As a consequence of Theorem 3.8, we get that $(V_Q, |Q|)$ belongs to $S$. Moreover, the following result shows that this property characterizes $\mathcal{Q}$:

**Proposition 3.11.** *Given $T \in L(\mathcal{H})$ with polar decomposition $T = V_T|T|$, it holds $T \in \mathcal{Q}$ if and only if $(V_T, |T|) \in S$.*

**Proof.** If $T \in \mathcal{Q}$, from Theorem 3.8, it follows that $(V_T, |T|) \in S$.
On the other hand, if $V_T|T|V_T = V_T$ then $T^2 = V_T|T|V_T|T| = V_T|T| = T$, so that $T \in \mathcal{Q}$. □

Very recently, much attention has been paid to the so-called Aluthge transform. This notion has been introduced by Aluthge [1] as a useful tool for studying generalized hyponormal operators. If $T \in L(\mathcal{H})$ has polar decomposition $T = V|T|$ then the Aluthge transform is $\widetilde{T}_{1/2} := |T|^{1/2}V|T|^{1/2}$ and, more generally, for $0 < \lambda < 1$, $\widetilde{T}_\lambda := |T|^{1-\lambda}V|T|^\lambda$. The Duggal-Porta transform corresponds to the extreme case $\lambda = 0$, i.e., $\widetilde{T}_0 = |T|V$. The reader is referred to [4,2,13] for many results on these notions.

It turns out that, for an oblique projection, all these transforms coincide:

**Proposition 3.12.** *If $Q \in \mathcal{Q}$ then for all $\lambda$, $0 \leqslant \lambda < 1$ it holds*

$$\widetilde{Q}_\lambda = P_{N(Q)^\perp}.$$

**Proof.** We prove the case $0 < \lambda < 1$; the case $\lambda = 0$ has been proven in Lemma 3.2. Observe first that every $\widetilde{Q}_\lambda$ is an oblique projection: in fact $\widetilde{Q}_\lambda^2 = (|Q|^{1-\lambda}V_Q|Q|^\lambda)(|Q|^{1-\lambda}V_Q|Q|^\lambda) = |Q|^{1-\lambda}V_Q|Q|V_Q|Q|^\lambda = |Q|^{1-\lambda}V_Q|Q|^\lambda = \widetilde{Q}_\lambda$, because $V_Q|Q|V_Q = V_Q$ (see Lemma 3.1). Obviously, $R(\widetilde{Q}_\lambda) = R(|Q|^{1-\lambda}V_Q|Q|^\lambda) \subseteq R(|Q|^{1-\lambda}) = N(Q)^\perp$, because, in general, $\overline{R(|T|^t)} = \overline{R(T^*)} = N(T)^\perp$ for $t > 0$.

On the other hand, from the definition $\widetilde{Q}_\lambda = |Q|^{1-\lambda}V_Q|Q|^\lambda$ we get $|Q|^\lambda\widetilde{Q}_\lambda|Q|^{1-\lambda} = |Q|V_Q|Q| = |Q|$, by Lemma 3.6, and therefore, since ${|Q|^\lambda}^\dagger|Q|^\lambda = P_{N(Q)^\perp} = |Q|^{1-\lambda}(|Q|^{1-\lambda})^\dagger$, we also get $\widetilde{Q}_\lambda P_{N(Q)^\perp} = P_{N(Q)^\perp}$. In particular, $N(Q)^\perp \subseteq R(\widetilde{Q}_\lambda)$; we conclude that $R(\widetilde{Q}_\lambda) = N(Q)^\perp$. But, obviously, $N(Q) \subseteq N(\widetilde{Q}_\lambda)$ and, using Lemma 2.1, we obtain $\widetilde{Q}_\lambda = P_{N(Q)^\perp}$ because both oblique projections have the same range and comparable nullspaces. □

**Remark 3.13.** Observe the identity $|Q|^\lambda V_Q^*|Q|^{1-\lambda} = |Q|^{1-\lambda}V_Q|Q|^\lambda$, which follows from the fact that $\widetilde{Q}_\lambda$ is an orthogonal projection.

## 4. On the Moore–Penrose inverse of an oblique projection

The next result is essentially due to Greville [10], who proved it for matrices, but part of it was proven by Penrose [16]. With the addition of a closedness hypothesis, his proof is still valid for Hilbert space operators.

**Theorem 4.1.** *If $Q \in L(\mathcal{H})$ is an oblique projection then $Q^\dagger = P_{N(Q)^\perp}P_{R(Q)}$. Conversely, if $\mathcal{M}$ and $\mathcal{N}$ are closed subspaces of $\mathcal{H}$ such that $P_\mathcal{M}P_\mathcal{N}$ has closed range, then $(P_\mathcal{M}P_\mathcal{N})^\dagger$ is the unique oblique projection with range $R(P_\mathcal{N}P_\mathcal{M})$ and nullspace $R(P_\mathcal{M}P_\mathcal{N})^\perp = N(P_\mathcal{N}P_\mathcal{M})$.*

**Proof.** If $Q^2 = Q$, then $Q^\dagger = Q^\dagger QQ^\dagger = Q^\dagger Q^2Q^\dagger = (Q^\dagger Q)(QQ^\dagger) = P_{N(Q)^\perp}P_{R(Q)}$.

Since $R(P_\mathcal{M}P_\mathcal{N})$ is closed, the operator $Y = (P_\mathcal{M}P_\mathcal{N})^\dagger$ is well defined. Observe that, by the properties of the Moore–Penrose inverse, $R((P_\mathcal{M}P_\mathcal{N})^\dagger) = R((P_\mathcal{M}P_\mathcal{N})^*) = R(P_\mathcal{N}P_\mathcal{M})$. Then $R(Y) \subseteq \mathcal{N}$. Since $R(P_\mathcal{N}P_\mathcal{M})$ is also closed, $Y^* = (P_\mathcal{N}P_\mathcal{M})^\dagger$ and $R(Y^*) = R(P_\mathcal{M}P_\mathcal{N}) \subseteq \mathcal{M}$. Thus $P_\mathcal{N}Y = Y$ and $P_\mathcal{M}Y^* = Y^*$, so that $Y^2 = (YP_\mathcal{M})(P_\mathcal{N}Y) = Y(P_\mathcal{M}P_\mathcal{N})Y = Y$, by one of the defining properties of $(P_\mathcal{M}P_\mathcal{N})^\dagger$. □

**Remark 4.2.** Observe that $R((P_{\mathcal{M}}P_{\mathcal{N}})^\dagger) = R(P_{\mathcal{N}}P_{\mathcal{M}}) = P_{\mathcal{N}}\mathcal{M}$ and $N((P_{\mathcal{M}}P_{\mathcal{N}})^\dagger) = R((P_{\mathcal{M}}P_{\mathcal{N}})^{\dagger *})^\perp = R(P_{\mathcal{M}}P_{\mathcal{N}})^\perp = (P_{\mathcal{M}}\mathcal{N})^\perp$ and the fact that $(P_{\mathcal{M}}P_{\mathcal{N}})^\dagger$ is an oblique projection implies

$$P_{\mathcal{N}}\mathcal{M} \dot{+} (P_{\mathcal{M}}\mathcal{N})^\perp = \mathcal{H}.$$

This means that the mapping $(\mathcal{M},\mathcal{N}) \longrightarrow (P_{\mathcal{N}}\mathcal{M}, P_{\mathcal{M}}\mathcal{N})$ sends a pair $(\mathcal{M},\mathcal{N})$ such that $\mathcal{M}+\mathcal{N}^\perp$ is closed into a pair $(P_{\mathcal{N}}\mathcal{M}, P_{\mathcal{M}}\mathcal{N})$ such that $P_{\mathcal{N}}\mathcal{M} \dot{+} (P_{\mathcal{M}}\mathcal{N})^\perp = \mathcal{H}$.

We prove now one of the main result of the section, namely, that the map $Q \longrightarrow Q^\dagger$ is Lipschitzian of constant 2.

**Theorem 4.3.** *Given $Q_1, Q_2 \in \mathcal{Q}$ it holds*

$$\|Q_1^\dagger - Q_2^\dagger\| \leqslant 2\|Q_1 - Q_2\|.$$

**Proof.** Recall a result by Kato, which states that $\|P_{R(Q_1)} - P_{R(Q_2)}\| \leqslant \|Q_1 - Q_2\|$ [14] (see also Mbekhta [15]). Then:

$$\begin{aligned}\|Q_1^\dagger - Q_2^\dagger\| &= \|P_{N(Q_1)^\perp}P_{R(Q_1)} - P_{N(Q_2)^\perp}P_{R(Q_2)}\|\\ &\leqslant \|P_{N(Q_1)^\perp}(P_{R(Q_1)} - P_{R(Q_2)})\| + \|(P_{N(Q_1)^\perp} - P_{N(Q_2)^\perp})P_{R(Q_2)}\|\\ &\leqslant \|P_{R(Q_1)} - P_{R(Q_2)}\| + \|P_{N(Q_1)^\perp} - P_{N(Q_2)^\perp}\| \leqslant 2\|Q_1 - Q_2\|\end{aligned}$$

because $\|P_{N(Q_1)^\perp}\| = \|P_{R(Q_2)}\| = 1$ and $\|P_{N(Q_1)^\perp} - P_{R(Q_2)^\perp}\| = \|P_{R(Q_1^*)} - P_{R(Q_2^*)}\| \leqslant \|Q_1^* - Q_2^*\| = \|Q_1 - Q_2\|$. $\square$

**Remark 4.4.** (1) The continuity of $Q \longrightarrow Q^\dagger$ follows from Apostol's result [5] that $T \longrightarrow P_{R(T)}$ is continuous on $\Gamma_\varepsilon = \{T : \gamma(T) \geqslant \varepsilon\}$ for any $\varepsilon > 0$ and the fact that for any $Q \in \mathcal{Q}$ it holds that $\gamma(Q) \geqslant 1$, which follows by multiplying $I \geqslant P_{R(Q)}$ at left by $Q$ and at right by $Q^*$. The continuity of $T \longrightarrow P_{N(T)}$ on the same set $\Gamma_\varepsilon$ is analogous and Greville's identity $Q^\dagger = P_{N(Q)^\perp}P_{R(Q)}$ completes the proof. However, the approach followed here gives the finer result $\|Q_1^\dagger - Q_2^\dagger\| \leqslant 2\|Q_1 - Q_2\|$.

(2) If $\mathcal{Q}^\dagger = \{Q^\dagger : Q \in \mathcal{Q}\}$ then $^\dagger : \mathcal{Q} \longrightarrow \mathcal{Q}^\dagger$ is a bijective continuous map. However, it is not a homeomorphism. Observe, for $\mathcal{H} = \mathcal{C}^2$, that the sequence of projections $Q_n = \begin{pmatrix} 1 & n \\ 0 & 0 \end{pmatrix}$ does not converge; however, it is easy to check that $Q_n^\dagger = \begin{pmatrix} (1+n^2)^{-1} & 0 \\ n(1+n^2)^{-1} & 0 \end{pmatrix}$ converges to the nullmatrix, which is its own Moore–Penrose inverse.

## 5. Partial isometries of oblique projections

Observe that the polar decomposition of an orthogonal projection $P$ is the trivial factorization $P = P^2$: in fact, $P$ is at the same time a positive operator and a partial isometry. However, for an oblique projection $Q$, the natural question arises about how special are both, the partial isometry $V_Q$ and $|Q|$. This section is devoted to the first case.

There are partial isometries $V$ for which $V \neq V_Q$ for all $Q$: in fact, if $V \neq I$ is an isometry then $N(V) = \{0\}$, and there is only one projection $Q$ such that $N(Q) = \{0\}$, namely, $Q = I$. Of course, the polar decomposition of $I$ is the trivial $I = I \cdot I$. Observe that even if $\dim \mathcal{H} < \infty$ not every partial isometry is contained in $\mathcal{J}_{\mathcal{Q}}$. Take, for instance, $V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on $\mathcal{H} = \mathcal{C}^2$.

In what follows we denote by $GL(\mathcal{H})$ the group of invertible bounded linear operators and by $GL(\mathcal{H})^+$ the subset of $GL(\mathcal{H})$ of positive operators. The next theorem characterizes the set $\mathcal{J}_{\mathcal{Q}}$:

**Theorem 5.1.** *For a partial isometry $V \in L(\mathcal{H})$ the following conditions are equivalent*:

1. *there exists $Q \in \mathcal{Q}$ such that $V = V_Q$, in fact $Q$ is uniquely determined as $Q = P_{R(V)//N(V)}$*;
2. $V|_{R(V)} \in GL(R(V))^+$;
3. *there exists $A \in L(\mathcal{H})^+$ such that $R(A) = R(V)$ and $V = AP_{N(V)^\perp}$*;
4. *there exists $\alpha > 0$ such that $V^2V^* \geqslant \alpha P_{R(V)}$.*

**Proof.** $1 \to 2$: If $V = V_Q$, for $Q \in \mathcal{Q}$, then $R(V) = R(Q)$ and $Q = |Q^*|V$, or $V = |Q^*|^\dagger Q$. Therefore, $VP_{R(V)} = VP_{R(Q)} = |Q^*|^\dagger QP_{R(Q)} = |Q^*|^\dagger P_{R(Q)} = |Q^*|^\dagger$ because $R(|Q^*|^\dagger) = R(|Q^*|) = R(Q)$; then $VP_{R(V)} = |Q^*|^\dagger$. This implies that $V|_{R(V)} = VP_{R(V)}|_{R(V)} = |Q^*|^\dagger|_{R(V)} \in GL(R(V))^+$.

$2 \to 1$: If $V|_{R(V)} \in GL(R(V))^+$ then $(VP_{R(V)})^\dagger VP_{R(V)} = P_{R(V)}$. Define $Q = (VP_{R(V)})^\dagger V$; it is easy to see that $Q = P_{R(V)} + (VP_{R(V)})^\dagger V(I - P_{R(V)})$ and then $Q^2 = Q$: in fact, $P_{R(V)}(VP_{R(V)})^\dagger V(I - P_{R(V)}) = (VP_{R(V)})^\dagger V(I - P_{R(V)})$ because $R((VP_{R(V)})^\dagger V(I - P_{R(V)})) \subset R(V)$; obviously, $(VP_{R(V)})^\dagger V(I - P_{R(V)}) P_{R(V)} = 0$ and $(VP_{R(V)})^\dagger V(I - P_{R(V)})(VP_{R(V)})^\dagger V(I - P_{R(V)}) = 0$ because $R((VP_{R(V)})^\dagger \subset R(V)$.

Since $(VP_{R(V)})^\dagger$ is positive and $R((VP_{R(V)})^\dagger) = R(V)$, it follows from the uniqueness of the polar decomposition that $(VP_{R(V)})^\dagger = |Q^*|$ and $V = V_Q$.

$2 \leftrightarrow 4$: $V|_{R(V)} \in GL(R(V))^+$ is equivalent to $V|_{R(V)} \geqslant \beta I$, on $R(V)$, for some $\beta > 0$; but observe that this is equivalent to $V^2V^* \geqslant \beta P_{R(V)}$.

$1 \to 3$ is proved in Theorem 3.8, 1.

To prove $3 \to 1$ suppose that there exists $A \in L(\mathcal{H})^+$ such that $V = AP_{N(V)^\perp}$ and $R(A) = R(V)$. Then $VV^* = AP_{N(V)^\perp}A = P_{R(V)}$ and $V^*V = P_{N(V)^\perp}A^2P_{N(V)^\perp} = P_{N(V)^\perp}$, because $V$ is a partial isometry. Let $Q = A^2P_{N(V)^\perp}$, then $Q^2 = Q$. Also, $QQ^* = A^2P_{N(V)^\perp}A^2 = AP_{R(V)}A = AP_{R(A)}A = A^2$, so that $|Q^*| = A$ and $V_Q = AP_{N(V)^\perp} = V$ because $R(Q) = R(|Q^*|) = R(A) = R(V)$ and $N(Q) = N(AV) = N(V)$. $\square$

We have just proved that

$$\mathcal{J}_\mathcal{Q} = \{V \in \mathcal{J} : V|_{R(V)} \in GL(R(V))^+\}.$$

Our next result shows that the correspondence between $Q$ and $V_Q$ is a homeomorphism between $\mathcal{Q}$ and $\mathcal{J}_\mathcal{Q}$.

**Theorem 5.2.** *The map*

$$\varphi : \mathcal{J}_\mathcal{Q} \longrightarrow \mathcal{Q}, \quad \varphi(V) := Q_V = (V^2V^*)^\dagger V$$

*is a homeomorphism, which is the inverse of the map $Q \longrightarrow V_Q$.*

**Proof.** Notice first that if $T \in L(\mathcal{H})$, then $T \longrightarrow TT^*$ and $T \longrightarrow T^*T$ are always continuous. In particular, if $V$ is a partial isometry, we get that $V \longrightarrow P_{R(V)} = VV^*$ and $V \longrightarrow P_{N(V)^\perp} = V^*V$, are continuous. But if $V \in \mathcal{J}_\mathcal{Q}$ then $\varphi(V) = P_{R(V)//N(V)} = P_{R(V)}(P_{R(V)} + P_{N(V)^\perp} - I)^{-1}P_{N(V)^\perp}$; the first equality has been proved in the last theorem, and the second follows by a well-known formula (see [17,6]); therefore, the continuity of $\varphi$ follows. On the other hand, the continuity of the inverse of $\varphi$ has been proved in Corollary 3.9. Also $|Q_V^*| = (V^2V^*)^\dagger$ and $V_{Q_V} = V$. Observe that if $V \in \mathcal{J}_\mathcal{Q}$ then $R(V) \dot{+} N(V) = \mathcal{H}$, which is not true in general for an arbitrary partial isometry. $\square$

## 6. Positive parts of oblique projections

In this section we characterize all (closed range) positive operators $A$ such that $A = |Q|$ for some $Q \in \mathcal{Q}$. Of course, such $A$ must satisfy $\gamma(A) \geqslant 1$. However, this condition is not sufficient. The next theorem describes the set $L(\mathcal{H})^+_\mathcal{Q}$:

**Theorem 6.1.** *Let $B \in L(\mathcal{H})^+$. There exists $Q \in \mathcal{Q}$ such that $|Q| = B$ if and only if $\gamma(B) \geqslant 1$ and $\dim \overline{R(B^2 - P_{R(B)})} \leqslant \dim N(B)$.*

**Proof.** By interchanging $Q$ and $Q^*$, we will study the equation $|Q^*| = B$. Suppose, then, that $B^2 = QQ^*$, so that $R(B^2)$ is closed and so is $R(B)$ and $R(B) = R(V)$. If the matrix representation of $Q$ along the decomposition $\mathcal{H} = R(B) \oplus N(B)$ is $Q = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$, where $a : N(B) \longrightarrow R(B),\ a = Q|_{N(B)}$, then $QQ^* = \begin{pmatrix} 1+aa^* & 0 \\ 0 & 0 \end{pmatrix}$ and $B^2|_{R(B)} = 1 + aa^*$. Therefore, $B^2 \geqslant P_{R(B)}$ and it is easy to see that therefore, $B \geqslant P_{R(B)}$ and $\gamma(B) \geqslant 1$. Also, $\dim \overline{R(B^2 - P_{R(B)})} = \dim \overline{R(aa^*)} = \dim \overline{R(a)} \leqslant \dim N(B)$, because since $a$ is a linear map from $N(B)$ to $R(B)$ we can conclude that $\dim \overline{R(a)} \leqslant \dim N(B)$.

Conversely, if $\gamma(B) \geqslant 1$ then $\gamma(B^2) \geqslant 1$ so that $B^2 - P_{R(B)}$ is positive. Let $D = (B^2 - P_{R(B)})^{1/2}$ and consider a subspace $\mathcal{S} \subseteq N(B)$ such that $\dim \mathcal{S} = \dim \overline{R(D)}$. This is possible because $\dim \overline{R(D)} = \dim \overline{R(B^2 - P_{R(B)})} \leqslant \dim N(B)$. If $U$ is a partial isometry with initial space $\mathcal{S}$ and final space $\overline{R(D)}$, then $DU(DU)^* = D^2 = B^2 - P_{R(B)}$. Hence, if $Q = P_{R(B)} + DU$, it follows that $Q^2 = Q$ and $QQ^* = P_{R(B)} + D^2 = B^2$, so that $B = |Q^*|$. □

In contrast with the case of partial isometries, which uniquely determine their corresponding oblique projections (see Section 5), the fibres of the maps $Q \longrightarrow |Q|$ and $Q \longrightarrow |Q^*|$ are not singletons. The following theorem characterizes the fibre $\{Q \in \mathcal{Q} : |Q^*| = B\}$, for $B \in L(\mathcal{H})^+_{\mathcal{Q}}$; the case of $\{Q \in \mathcal{Q} : |Q| = B\}$ is analogous.

**Theorem 6.2.** *Consider $B \in L(\mathcal{H})^+_{\mathcal{Q}}$. For $Q \in \mathcal{Q}$ the following conditions are equivalent:*

1. $|Q^*| = B$;
2. $Q = P_{R(B)} + (B^2 - P_{R(B)})^{1/2}U$, *where $U \in \mathcal{J}$ has final space $\overline{R(B^2 - P_{R(B)})}$ and initial space contained in $N(B)$;*
3. $V_Q = B^\dagger + (P_{R(B)} - B^{2^\dagger})^{1/2}U$, *where $U \in \mathcal{J}$ has final space $\overline{R(B^2 - P_{R(B)})}$ and initial space contained in $N(B)$.*

**Proof.** $1 \longrightarrow 2$ follows from the proof of Theorem 6.1.

$2 \longrightarrow 3$: if $Q = P_{R(B)} + (B^2 - P_{R(B)})^{1/2}U$ then $QQ^* = B$ because $UU^* = P_{\overline{R(B^2 - P_{R(B)})}}$. Therefore $V_Q = B^\dagger Q = B^\dagger(P_{R(B)} + (B^2 - P_{R(B)})^{1/2}U) = B^\dagger + (P_{R(B)} - B^{2^\dagger})^{1/2}U$.

$3 \longrightarrow 1$: Observe first that $V_Q V_Q^* = P_{R(B)}$ so that $R(V_Q) = R(B)$. From the proof of $1 \longrightarrow 2$ of Theorem 5.1 it follows that $V_Q P_{R(V_Q)} = |Q^*|^\dagger$. In this case $|Q^*|^\dagger = V_Q P_{R(V_Q)} = V_Q P_{R(B)} = B^\dagger$ so that $|Q^*| = B$. □

The next result characterizes the image $\mathcal{L}$, in $L(\mathcal{H})^+ \times L(\mathcal{H})^+$, of the map $Q \longrightarrow (QQ^*, Q^*Q)$. Observe that this is related to a paper of Horn and Olkin [12] about the relationship between $AA^*$ and $A^*A$, for a matrix $A$.

**Theorem 6.3.** *Let $A, B \in L(\mathcal{H})^+$ with a closed range. Then, there exists $Q \in \mathcal{Q}$ such that $|Q| = A^{1/2}$ and $|Q^*| = B^{1/2}$ if and only if $P_{R(A)}BP_{R(A)} = P_{R(A)}$ and $P_{R(B)}AP_{R(B)} = P_{R(B)}$.*

**Proof.** If $QQ^* = B$ and $Q^*Q = A$ then $R(Q) = R(B)$ and $N(Q) = N(A)$, or equivalently, $Q = P_{R(B)//N(A)}$. Applying Theorem 3.8(5) we get that $Q = BP_{R(A)} = P_{R(B)}A$. Therefore $P_{R(A)}BP_{R(A)} = P_{R(A)}Q = P_{R(A)}$ because $P_{R(A)}$ and $Q$ have the same nullspace; in the same way, $P_{R(B)}AP_{R(B)} = QP_{R(B)} = P_{R(B)}$ because $Q$ and $P_{R(B)}$ have the same range.

Conversely, suppose that $P_{R(A)}BP_{R(A)} = P_{R(A)}$ and consider $Q = BP_{R(A)}$. It follows that $Q$ is idempotent. To compute the nullspace of $Q$ observe that

$$N(A) = N(P_{R(A)}) = N(P_{R(A)}BP_{R(A)}) = N(B^{1/2}P_{R(A)}) = R(A) \cap N(B) \dotplus N(A).$$

Therefore $R(A) \cap N(B) = \{0\}$ and $N(P_{R(A)}BP_{R(A)}) = N(A)$. Then $N(Q) = N(BP_{R(A)}) = N(B^{1/2}P_{R(A)}) = N(A)$. Observe that $R(Q) = B(R(A))$. In a similar way, from $P_{R(B)}AP_{R(B)} = P_{R(B)}$ we get that $R(B) \cap N(A) = \{0\}$ so that $\mathcal{H} = R(Q) \dotplus N(Q) = B(R(A)) \dotplus N(A) \subseteq R(B) \dotplus N(A)$. This implies that $R(Q) = B(R(A)) = R(B)$. Hence $Q = P_{R(B)//N(A)}$. To see that $QQ^* = B$ observe that multiplying both sides of the equality $P_{R(A)}BP_{R(A)} = P_{R(A)}$ by $B^{1/2}$ it follows that $B^{1/2}P_{R(A)}B^{1/2}$ is an orthogonal projection, in fact $B^{1/2}P_{R(A)}B^{1/2} = P_{R(B)}$. Then $QQ^* = BP_{R(A)}B = B$.

In the same way, using that $P_{R(B)}AP_{R(B)} = P_{R(B)}$, $\widetilde{Q} = AP_{R(B)}$ is an oblique projection such that $R(\widetilde{Q}) = R(A)$, $N(\widetilde{Q}) = N(B)$ and $\widetilde{Q}\widetilde{Q}^* = A$. Therefore $\widetilde{Q} = P_{R(A)//N(B)}$ so that $\widetilde{Q} = Q^*$, which shows that $Q^*Q = \widetilde{Q}\widetilde{Q}^* = A$. $\square$

**Corollary 6.4.** *The inverse of the map* $Q \longrightarrow (QQ^*, Q^*Q)$, *for* $Q \in \mathcal{Q}$, *is given by* $(B, A) \longrightarrow BP_{R(A)} (= P_{R(B)}A)$, *for* $(B, A) \in \mathcal{L}$.

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